Optimization of functionals

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Problem set 1. Calculus of variations

Solutions

1. Suppose the Lagrangian $L(t, x, x')$ and $x(t)$ are of class C^2. Show that the EL equation can be written as

$$x'' D_{33}^2 L + x' D_{32}^2 L + D_{31}^2 L - D_2 L = 0.$$

Solution. It follows from the Chain rule, since

$$\frac{d}{dt} D_3 L = x'' D_{33}^2 L + x' D_{32}^2 L + D_{31}^2 L.$$

2. Consider a Lagrangian $L(x, x')$ which does not depend explicitly on t. If L is of class C^1 and there exists x'', show that the EL equation becomes

$$L - x' \frac{\partial L}{\partial x'} = C,$$

where C is a constant.

Solution. Notice that $\frac{d}{dt}[L - x' \frac{\partial L}{\partial x'}] = x'[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x'}]$, hence the conclusion follows from the EL equation.

Remark. By assuming that x are of class C^1, the differentiability of x' (that is, the existence of x'') can be proved (see p. 25 in O. Bolza, *Lectures on the calculus of variations*, Chelsea, New York, 1961).

3. Find the function $\hat{x} : [0, T] \to \mathbb{R}$ of class C^1 that minimizes the functional

$$\int_0^T [x^2(t) + cx'(t)^2] dt, \qquad x(0) = x_0, \; x(T) = 0,$$

where c is a positive constant.

Solution. The EL equation takes the form $x'' - x/c = 0$ whose general solution is

$$\hat{x}(t) = k_1 e^{rt} + k_2 e^{-rt},$$

where $r^2 c = 1$. By using the conditions $x(0) = x_0$ and $x(T) = 0$, we find

$$k_1 = \frac{-x_0 e^{-rT}}{e^{rT} - e^{-rT}}$$

and $k_2 = x_0 - k_1$. Since the Lagrangian is convex, $\hat{x}$ is indeed a minimizer.

4. Consider the functional

$$T(f) := \int_0^{b_1} \sqrt{\frac{1 + [f'(x)]^2}{-2gf(x)}} dx, \qquad f(0) = 0, \; f(b_1) = b_2.$$

Show that the associated EL equation[1] becomes

$$f(x)\left(1+[f'(x)]^2\right)=c, \qquad f(0)=0,\ f(b_1)=b_2. \tag{1}$$

Solution. From Exercise 2, the EL equation takes the form

$$L(f,f')-f'\frac{1}{2L(f,f')}\cdot\frac{f'}{-gf}=C,$$

where $L(f,f')=\sqrt{\frac{1+(f')^2}{-2gf}}$. That is,

$$\sqrt{\frac{1+(f')^2}{-f}}-\frac{(f')^2}{\sqrt{1+(f')^2}\sqrt{-f}}=C\sqrt{2g}.$$

Use the algebraic equality

$$\sqrt{1+d^2}-\frac{d^2}{\sqrt{1+d^2}}=\frac{1}{\sqrt{1+d^2}}$$

and let $c=-1/(2gC^2)$.

5. Prove that the following parametric curve is a solution to (1)

$$\begin{aligned} x(\alpha) &= R(\alpha-\sin\alpha) && (2)\\ y(\alpha) &= -R(1-\cos\alpha), \qquad 0\le\alpha\le\alpha_1 && (3)\end{aligned}$$

where $R=-b_2/(1-\cos\alpha_1)$ and α_1 is a solution, in the interval $(0,2\pi)$, to

$$\frac{\alpha-\sin\alpha}{1-\cos\alpha}=-\frac{b_1}{b_2}.$$

[1]Ignore the assumptions required to obtain the EL equation

Solution. Notice that (2) can be written as $R(\alpha - \sin\alpha) - x = 0$. By using the implicit function theorem, α can be written as $\alpha(x)$ in the interval $(0, 2\pi)$, and

$$R[\alpha'(x) - \cos(\alpha(x)) \cdot \alpha'(x)] - 1 = 0.$$

That is, $\alpha'(x) = \frac{1}{R[1-\cos(\alpha(x))]}$. By the Chain rule

$$y'(x) = y'(\alpha(x)) \cdot \alpha'(x) = \frac{-\operatorname{sen}(\alpha(x))}{1 - \cos(\alpha(x))}.$$

We can verify that $y[1 + (y')^2] = -2R$ as required. Further, the conditions $x(\alpha_1) = b_1$ and $y(\alpha_1) = b_2$ yield

$$\frac{\alpha_1 - \sin\alpha_1}{1 - \cos\alpha_1} = -\frac{b_1}{b_2} \tag{4}$$

and $R = -b_2/(1 - \cos\alpha_1)$. The solution α_1 to equation (4) exists because the function

$$g(\alpha) := \frac{\alpha - \sin\alpha}{1 - \cos\alpha}, \qquad 0 < \alpha < 2\pi,$$

is continuous, positive, $\text{lím}_{\alpha \to 2\pi^-} g(\alpha) = \infty$, and $\text{lím}_{\alpha \to 0^+} g(\alpha) = 0$. The latter limit can be verified by using L'Hôpital's rule (twice).