Optimization of functionals

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Problem set 1

1. Let $f, g : S \to \mathbb{R}$ be convex functions, where $S \subseteq \mathbb{R}^n$ is convex. Show the following:

   (a) If c is a nonnegative real number, then $f + cg$ is convex.

   (b) Let $J \subseteq \mathbb{R}$ be an interval such that $f(x) \in J$ for every $x \in S$. If $F : J \to \mathbb{R}$ is convex and increasing, then $F \circ f$ is convex.

2. Show that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its **epigraph**

$$\{(x, y) \in \mathbb{R}^{n+1} \mid y \geq f(x)\}$$

   is a convex set.

3. Show that $f(x) = |x|$ is convex on $\mathbb{R}^n$. Is f strictly convex? What about $g(x) = |x|^2$?

4. Show that the set of global minimizers (which could be empty) of any convex function is convex. Prove also that strictly convex functions have at most one global minimizer.

5. Let $f_n : \mathbb{R}^n \to \mathbb{R}$ be a convex function for each $n \in \mathbb{N}$. Show the following assertions.

   (a) If (f_n) converges (pointwise) to f, then f is convex.

   (b) If $F(x) := \sup_{n \geq 1} f_n(x)$ is finite for each $x \in J$, then F is convex.

6. **(Least squares)** Let $A \in \mathcal{M}_{m \times n}$, with $m > n$, and $b \in \mathbb{R}^m$. The system $Ax = b$ usually does not have a solution $x \in \mathbb{R}^n$, then an alternative is to find the *least-squares solution* $\hat{x}$—if it exists—, that is,

$$|A\hat{x} - b|^2 = \min_{x \in \mathbb{R}^n} |Ax - b|^2.$$

   Assume $\text{rank}(A) = n$ and prove that there exists a unique global minimizer $\hat{x}$, given by

$$\hat{x} = (A^\top A)^{-1} A^\top b.$$

   Hint: Since rank$(A) = n$, then $A^\top A$ is invertible.

7. Let $a \in \mathbb{R}^n$, $a \neq 0$. Find the unique solution to the problem

$$\max_{x \in \mathbb{R}^n} \{a^\top x : |x|^2 = 1\}.$$

   Hint: Cauchy-Schwarz inequality could be useful.

8. **(Spectral theorem)** Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix.

   (a) Use Lagrange multipliers to show that there exists $\lambda_1 \in \mathbb{R}$ and $u_1 \in \mathbb{R}^n$, $|u_1| = 1$, such that
   $$Au_1 = \lambda_1 u_1$$
   and
   $$x \in \mathbb{R}^n, \ |x| = 1 \quad \Rightarrow \quad x^\top Ax \geq \lambda_1. \tag{1}$$

   (b) Show that λ_1 is the smallest eigenvalue of A.

   (c) Show that exists $\lambda_2 \in \mathbb{R}$ and $u_2 \in \mathbb{R}^n$, $|u_2| = 1$, such that
   $$Au_2 = \lambda_2 u_2$$
   and
   $$u_2^\top u_1 = 0.$$
   Hint: Consider $W_1 = \{x \in \mathbb{R}^n \mid x^\top u_1 = 0\}$, verify that $Ax \in W_1$ for every $x \in W_1$, and find a minimizer u_2 of $x^\top Ax$ in some compact subset of W_1.

   (d) Prove that there exist an orthonormal basis $\{u_1, \ldots, u_n\}$ of $\mathbb{R}^n$ and a vector $(\lambda_1, \ldots, \lambda_n)^\top$ such that
   $$Au_j = \lambda_j u_j, \quad 1 \leq j \leq n.$$

9. Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix. Prove the following:

   (a) $\operatorname{tr}(A) := \sum_{j=1}^n A_{jj} = \sum_{j=1}^n \lambda_j$.
   Hint: Recall Exercise 1.8(d) to show that $AU = U\Lambda$, where Λ is diagonal and the columns of U are eigenvectors.

   (b) A is positive semidefinite if and only if its eigenvalues are nonnegative.

   (c) A is positive definite if and only if its eigenvalues are positive.

10. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Consider the problem
   $$\text{ínf}\{h(x) \mid x \in \mathbb{R}^n\}.$$
   In numerical analysis, a vector $v \in \mathbb{R}^n \setminus \{0\}$ is said to be a *descent direction* of f at a if $D_v h(a) < 0$. If $\nabla h(a) \neq 0$, then $-\nabla h(a)$ is called the *steepest-descent direction* of h at a. Justify these names by proving the following:

   (a) If $D_v h(a) < 0$, then there exists $t_0 > 0$ such that
   $$h(a + tv) < h(a) \qquad \forall t \in (0, t_0].$$

   (b) There exists a solution to
   $$\text{mín}_{v \in \mathbb{R}^n}\{D_v h(a) : |v|^2 = 1\},$$
   and such a solution is given by $-\nabla h(a)/|\nabla h(a)|$.

11. Consider the optimization problem

$$P = \begin{cases} min \quad \frac{1}{3}\sum_{i=1}^{n} x_i^3 \quad s.t. \\ \sum_{i=1}^{n} x_i = 0, \\ \sum_{i=1}^{n} x_i^2 = n. \end{cases}$$

Find a global minimizer and a global maximizer.

12. Consider the optimization problem

$$P = \begin{cases} max \quad x_1^3 + x_2^3 + \cdots + x_n^3 \quad s.t. \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \end{cases}$$

Determine the global maximizers of the problem P; then prove the inequality

$$\sum_{i=1}^{n} |x_i|^3 \leq \left(\sum_{i=1}^{n} |x_i|^2 \right)^{3/2} \qquad \forall (x_1, \ldots, x_n) \in \mathbb{R}^n.$$