Optimization of functionals

Department of Mathematics
Cinvestav-IPN

Problem set 2. Lower semicontinuous functions

2.1 Let $\mathcal{F}$ be a collection of lsc functions on the metric space (X,d). Show that

$$F(x) := \sup\{f(x) \mid f \in \mathcal{F}\}, \qquad x \in X,$$

is lsc.

Hint: Notice that $\{x \in X \mid F(x) > \alpha\} = \cup_{f\in\mathcal{F}}\{x \in X \mid f(x) > \alpha\}$.

2.2 Let $f, g : X \to (-\infty, \infty]$ be lsc functions on the metric space (X,d). If $r > 0$, then show that rf and $f+g$ are lsc.

2.3 Prove that $f : X \to \mathbb{R}$ is continuous if and only if f is both l.s.c. and u.s.c.

2.4 Let A be a subset of the metric space (X,d). Consider the function

$$I_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{if } x \notin A. \end{cases}$$

Show that I_A is lsc if and only if A is closed.

2.5 Let (X,d) be a metric space and $\emptyset \neq A \subseteq X$. Define

$$d(x,A) := \inf_{a\in A} d(x,a) \quad x \in X. \tag{1}$$

Prove the following:

a) $d(x,A) \leq d(x,y) + d(y,A)$ for every $x, y \in X$,

b) the function $d(\cdot, A) : X \to \mathbb{R}$ is uniformly continuous,

c) if A is closed and $d(x,A) = 0$, then $x \in A$.

2.6 Let (X,d) be a metric space, $f : X \to \mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$.

a) Give an example of lsc functions f and g such that $g \circ f$ is not lsc.

b) Suppose f is continuous and g is lsc. Prove that $g \circ f$ is lsc.

2.7 (**The Fundamental Theorem of Algebra**). Let $p(z) = a_n z^n + \ldots + a_1 z + a_0$ be a polynomial with complex coefficients, $a_n \neq 0$ and $n \geq 1$. Define the function $f(z) := |p(z)|$ for each $z \in \mathbb{C}$.

(a) Show that f has a global minimizer.

Hint: Show that f *is coercive.*

(b) Find explicitly one (there could be more) global minimizer of f when (i) $p(z) = a_1 z + \ldots + a_n z^n$, that is $a_0 = 0$, and (ii) $p(z) = a_0 + a_k z^k$ with $a_k \neq 0$.

(c) Let $z_0 \in \mathbb{C}$. Explain why there exist complex numbers $c_0, c_1, \ldots, c_n$ such that
$$p(z) = c_0 + c_1(z - z_0) + \ldots + c_n(z - z_0)^n.$$

Hint: Write $p(z) = p((z - z_0) + z_0)$.

Further, prove that, for some $k = 1, \ldots, n$,
$$p(z) = c_0 + c_k(z - z_0)^k + (z - z_0)^{k+1} q(z),$$
where $c_k \neq 0$ and q is a polynomial.

(d) Let z_0 be a global minimizer of f, $t \in (0, 1)$, and $w \in \mathbb{C}$ satisfies $c_0 + c_k w^k = 0$. Suppose $f(z_0) > 0$, that is, $c_0 \neq 0$. Show that
$$f(z_0 + tw) \leq |c_0|(1 - t^k) + |tw|^{k+1}|q(z_0 + tw)|$$
and
$$t|w^{k+1} q(z_0 + tw)| < |c_0|$$
for some t small enough.

(e) Prove the Fundamental Theorem of Algebra.

2.8 (**Baby EVP**) Let X be a finite-dimensional vector space. Suppose $f : X \to \mathbb{R}$ is lsc and bounded below. Let $\varepsilon > 0$ and $x_0 \in X$ satisfy
$$f(x_0) \leq \text{ínf}\, f + \varepsilon. \tag{2}$$

Prove (without using Ekeland's variational principle!) that there exists $\overline{x} \in X$ such that

(i) $f(\overline{x}) \leq f(x_0)$,

(ii) $|\overline{x} - x_0| \leq \sqrt{\varepsilon}$, and

(iii) $f(\overline{x}) \leq f(x) + \sqrt{\varepsilon}|x - \overline{x}|$ for all $x \in X$.

In order to accomplish the proof, proceed as follows:

(a) Show that $g(x) = f(x) + \sqrt{\varepsilon}|x - x_0|$ has a global minimizer $\overline{x}$ in X.

Hint: Show that g is coercive and lsc.

(b) Use the inequality $g(\overline{x}) \leq g(x_0)$ to prove (i) and (ii).

(c) Finally, use (a) to prove (iii).

Hint: Notice that $|x - x_0| \leq |x - \overline{x}| + |\overline{x} - x_0|$.

2.9 Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and bounded from below. Assume (x_n) is a minimizing sequence for f. Prove that there exists another sequence (y_n) such that
$$f(y_n) \leq f(x_n), \quad |x_n - y_n| \to 0, \quad \text{and} \quad \nabla f(y_n) \to 0.$$

Hint: If $\varepsilon_n := f(x_n) - \text{ínf} f > 0$, then apply Baby EVP. The following inequality would be useful

$$\frac{1}{t}[f(y_n + ty) - f(y_n)] \geq -\sqrt{\varepsilon_n}$$

for $t > 0$ and $|y| = 1$.

2.10 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function such that, for some $a > 0$ and $b \in \mathbb{R}$,

$$f(x) \geq a|x| + b \qquad \forall x \in \mathbb{R}^n.$$

Prove the that the image $\nabla f(\mathbb{R}^n)$ is dense in the ball $\{x \in \mathbb{R}^n : |x| \leq a\}$.

Hint: Consider the function $g(x) = f(x) - \langle x, u \rangle$, where $|u| \leq a$.

2.11 Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Suppose $g : \Omega \times \mathbb{R} \to [0, \infty)$ is measurable and

$$x \mapsto g(\omega, x)$$

is lsc for each $\omega \in \Omega$. Define the functional

$$J(x) = \int_\Omega g(\omega, x(\omega))d\mu, \qquad x \in L^p(\Omega),$$

where $1 < p < \infty$. Show that J is lsc on $L^p(\Omega)$.

Hint: Pick $x_n \to \overline{x}$ in L^p. Extract a subsequence (x_{n_k}) such that

$$\lim_{k\to\infty} J(x_{n_k}) = \liminf_{n\to\infty} J(x_n).$$

Is there a subsequence of (x_{n_k}) that converges a.e. to $\overline{x}$?

2.12 Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable. Suppose

$$x \mapsto g(\omega, x), \qquad \omega \in \Omega,$$

is lsc and

$$g(\omega, x) \geq -a(\omega) - cx^p \qquad \forall (\omega, y) \in \Omega \times \mathbb{R}$$

for some $a \in L^1(\Omega)$, $c \in \mathbb{R}$, and $1 < p < \infty$. Show that

$$J(x) = \int_\Omega g(\omega, x(\omega))d\mu, \qquad x \in L^p(\Omega),$$

is lsc.

2.13 (**The length functional**) Let C^0 be the set of continuous functions from the interval $[a, b]$ to the metric space (X, d). Endow C^0 with the metric

$$\rho(\phi, \psi) := \sup_{a\leq t\leq b} d(\phi(t), \psi(t)), \qquad \phi, \psi \in C^0.$$

(a) Given a partition $P = \{t_0 = a < t_1 < \cdots < t_N = b\}$ of $[a, b]$, let

$$V_P(\phi) = \sum_{i=0}^{N-1} d(\phi(t_i), \phi(t_{i+1})), \quad \phi \in C^0.$$

Show that V_P is continuous on the metric space (C^0, ρ).

(b) Let $l : C^0 \to \mathbb{R} \cup \{+\infty\}$ be given by

$$l(\phi) = \sup_P V_P(\phi),$$

where the supremum is taken over all the partitions of $[a, b]$. Give an example of a sequence (ϕ_n) in C^0 such that $\phi_n \to \phi$ but

$$l(\phi_n) \not\to l(\phi).$$

That is, in general, l **is not** continuous.

(c) Use (a) and Problem 2.1 to show that l is lower semicontinuous.