Optimization of functionals

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Problem set 1. Calculus of variations

DUE: September 5, 2024

1. Suppose the Lagrangian $L(t,x,x')$ and $x(t)$ are of class C^2. Show that the EL (Euler–Lagrange) equation can be written as

$$x'' D^2_{33}L + x' D^2_{32}L + D^2_{31}L - D_2 L = 0.$$

2. Consider a Lagrangian $L(x,x')$ which does not depend explicitly on t. Show that the EL equation becomes

$$L - x' \frac{\partial L}{\partial x'} = C,$$

where C is a constant.

3. Find the function $\hat{x} : [0,T] \to \mathbb{R}$ of class C^1 that minimizes the functional

$$\int_0^T [x^2(t) + c x'(t)^2] dt, \qquad x(0) = x_0, \ x(T) = 0,$$

where c is a positive constant.

Answer: $\hat{x}(t) = \frac{x_0}{e^{rT} - e^{-rT}}[e^{r(T-t)} - e^{-r(T-t)}]$, *with* $r = 1/\sqrt{c}$.

4. Consider the functional

$$T(f) := \int_0^{b_1} \sqrt{\frac{1 + [f'(x)]^2}{-2gf(x)}} dx, \qquad f(0) = 0, \ f(b_1) = b_2.$$

Show that the associated EL equation[1] becomes

$$f(x)\left(1 + [f'(x)]^2\right) = c, \qquad f(0) = 0, \ f(b_1) = b_2. \tag{1}$$

5. Prove that the following parametric curve is a solution to (1)

$$\begin{aligned} x(\alpha) &= R(\alpha - \sin\alpha) \\ y(\alpha) &= -R(1 - \cos\alpha), \qquad 0 \le \alpha \le \alpha_1 \end{aligned}$$

where $R = -b_2/(1 - \cos\alpha_1)$ and α_1 is the unique solution, in the interval $(0, 2\pi)$, to

$$\frac{\alpha - \sin\alpha}{1 - \cos\alpha} = -\frac{b_1}{b_2}.$$

[1] Ignore the assumptions required to obtain the EL equation